Nonlinear Control

Lecture # 4

Stability of Equilibrium Points

Basic Concepts

$$\dot{x} = f(x)$$

f is locally Lipschitz over a domain $D \subset R^n$

Suppose $\bar{x} \in D$ is an equilibrium point; that is, $f(\bar{x}) = 0$

Characterize and study the stability of $\bar{x}$

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. No loss of generality

$$y = x - \bar{x}$$

$$\dot{y} = \dot{x} \; = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

Definition 3.1

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \ \forall\, t \geq 0$$

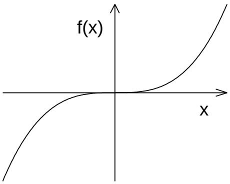
- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

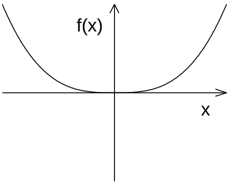
Scalar Systems ($n = 1$)

The behavior of $x(t)$ in the neighborhood of the origin can be determined by examining the sign of $f(x)$

The ε–δ requirement for stability is violated if $xf(x) > 0$ on either side of the origin



Unstable



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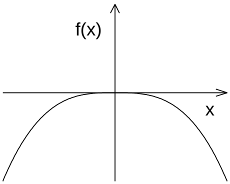
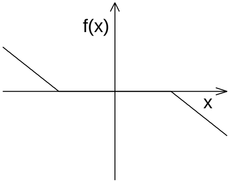
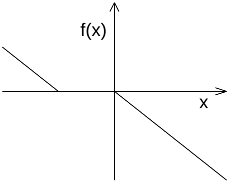


Unstable

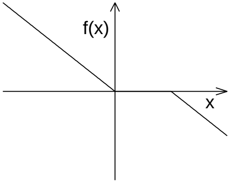
The origin is stable if and only if $xf(x) \leq 0$ in some neighborhood of the origin



Stable

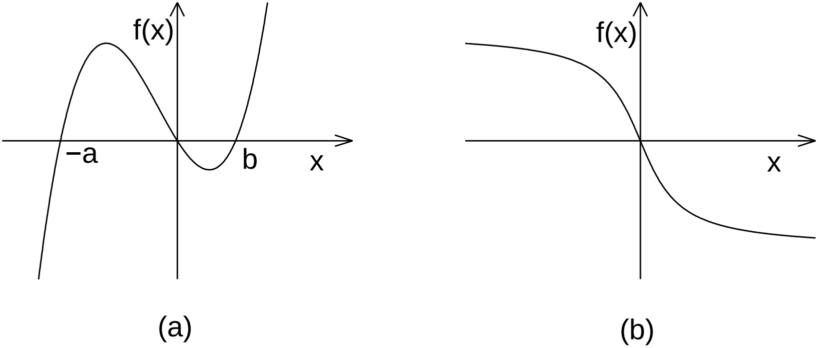


Stable



Stable

The origin is asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin



Asymptotically Stable

Globally Asymptotically Stable

Definition 3.2

Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is a locally Lipschitz function defined over a domain $D \subset R^n$ ($0 \in D$)

- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of

$$\dot{x} = f(x), \qquad x(0) = x_0$$

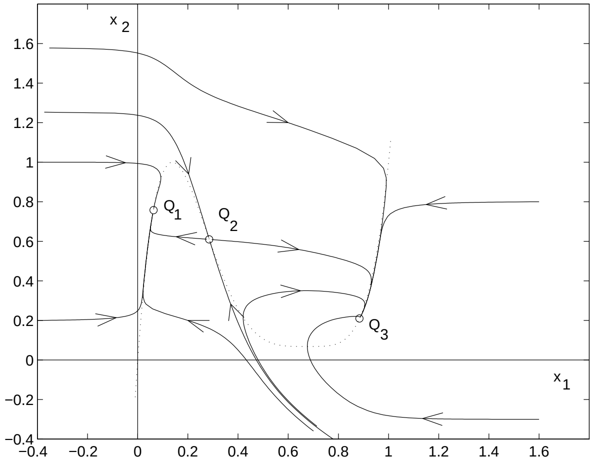
  is defined for all $t \geq 0$ and converges to the origin as t tends to infinity

- The origin is globally asymptotically stable if the region of attraction is the whole space R^n

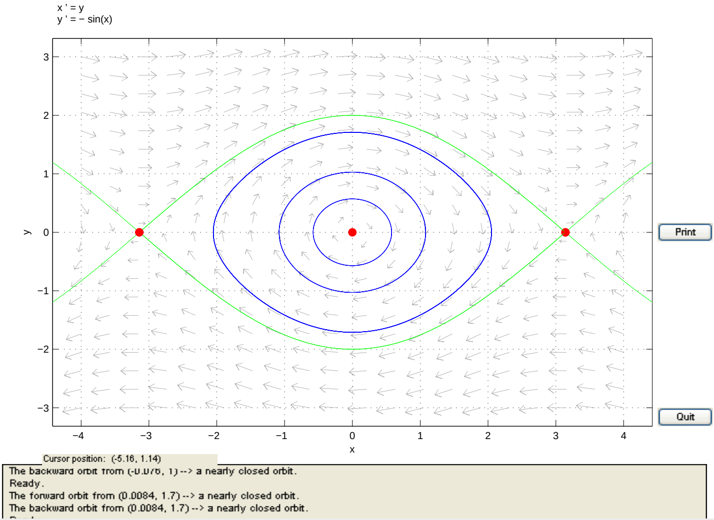
Two-dimensional Systems($n = 2$)

Type of equilibrium point	Stability Property
Center	
Stable Node	
Stable Focus	
Unstable Node	
Unstable Focus	
Saddle	

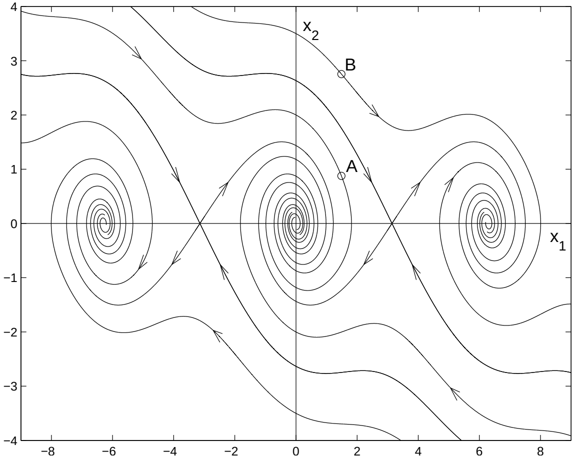
Example: Tunnel Diode Circuit



Example: Pendulum Without Friction



Example: Pendulum With Friction



Linear Time-Invariant Systems

$$\dot{x} = Ax$$

$$x(t) = \exp(At)x(0)$$

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik}$$

m_i is the order of the Jordan block J_i

$$\mathrm{Re}[\lambda_i] < 0 \ \ \forall\, i \quad \Leftrightarrow \quad \text{Asymptotically Stable}$$

$$\mathrm{Re}[\lambda_i] > 0 \ \ \text{for some } i \quad \Rightarrow \quad \text{Unstable}$$

$$\mathrm{Re}[\lambda_i] \le 0 \ \ \forall\, i \ \&\ m_i > 1 \text{ for } \ \mathrm{Re}[\lambda_i] = 0 \ \Rightarrow \quad \text{Unstable}$$

$$\mathrm{Re}[\lambda_i] \le 0 \ \ \forall\, i \ \&\ m_i = 1 \text{ for } \ \mathrm{Re}[\lambda_i] = 0 \ \Rightarrow \quad \text{Stable}$$

If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i, then the Jordan blocks of λ_i have order one if and only if $\mathrm{rank}(A - \lambda_i I) = n - q_i$

Theorem 3.1

The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

When all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix*

When the origin of a linear system is asymptotically stable, its solution satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\ t \geq 0, \quad k \geq 1,\ \lambda > 0$$

Exponential Stability

Definition 3.3

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\ t \geq 0$$

$k \geq 1,\ \lambda > 0$, for all $\|x(0)\| < c$

It is globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Exponential Stability $\Rightarrow$ Asymptotic Stability

Example 3.2

$$\dot{x} = -x^3$$

The origin is asymptotically stable

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

$x(t)$ does not satisfy $|x(t)| \le ke^{-\lambda t}|x(0)|$ because

$$|x(t)| \le ke^{-\lambda t}|x(0)| \;\Rightarrow\; \frac{e^{2\lambda t}}{1 + 2tx^2(0)} \le k^2$$

Impossible because $\displaystyle \lim_{t\to\infty} \frac{e^{2\lambda t}}{1 + 2tx^2(0)} = \infty$

Linearization

$$\dot{x} = f(x), \qquad f(0) = 0$$

f is continuously differentiable over $D = \{\|x\| < r\}$

$$J(x) = \frac{\partial f}{\partial x}(x)$$

$$h(\sigma) = f(\sigma x) \ \text{ for } \ 0 \le \sigma \le 1, \qquad h'(\sigma) = J(\sigma x)x$$

$$h(1) - h(0) = \int_0^1 h'(\sigma)\ d\sigma, \qquad h(0) = f(0) = 0$$

$$f(x) = \int_0^1 J(\sigma x)\ d\sigma\ x$$

$$f(x) = \int_0^1 J(\sigma x)\ d\sigma\ x$$

Set $A = J(0)$ and add and subtract Ax

$$f(x) = [A + G(x)]x, \text{ where } G(x) = \int_0^1 [J(\sigma x) - J(0)]\ d\sigma$$

$$G(x) \to 0 \quad \text{as} \quad x \to 0$$

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$

Theorem 3.2

- The origin is exponentially stable if and only if $\text{Re}[\lambda_i] < 0$ for all eigenvalues of A
- The origin is unstable if $\text{Re}[\lambda_i] > 0$ for some i

Linearization fails when $\text{Re}[\lambda_i] \leq 0$ for all i, with $\text{Re}[\lambda_i] = 0$ for some i

Example 3.3

$$\dot{x} = ax^3$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

Stable if $a = 0$; Asymp stable if $a < 0$; Unstable if $a > 0$

When $a < 0$, the origin is not exponentially stable